

CLASSIFICATION OF A FAMILY OF COMPLETELY TRANSITIVE CODES

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ABSTRACT. The *completely regular codes* in Hamming graphs have a high degree of combinatorial symmetry and have attracted a lot of interest since their introduction in 1973 by Delsarte. This paper studies the subfamily of *completely transitive codes*, those in which an automorphism group is transitive on each part of the distance partition. This family is a natural generalisation of the binary completely transitive codes introduced by Solé in 1990. We take the first step towards a classification of these codes, determining those for which the automorphism group is faithful on entries.

1. INTRODUCTION

Completely regular codes have been studied extensively ever since Delsarte [8] introduced them as a generalisation of perfect codes in 1973. Not only are these codes of interest to coding theorists as they possess a high degree of combinatorial symmetry, but, due to a result by Brouwer et al. [3, p.353], they are also the building blocks of certain types of distance regular graphs. At present there is no general classification of completely regular codes. However, certain families of completely regular codes have been characterised. For example, the first and third authors proved that certain binary completely regular codes are uniquely determined by their length and *minimum distance* [13, 14]. Borges et al. have classified all linear completely regular codes that have *covering radius* $\rho = 2$ and an *antipodal* dual code [2], showing that these codes are extensions of linear completely regular codes with covering radius $\rho = 1$. They also classified this later family of codes, and proved that these codes are in fact *coset-completely transitive*, a family of linear completely regular codes that were first introduced in the binary case by Solé [25] and then over an arbitrary finite field by the second and third authors [16]. In [1], Borges et al. classified binary coset-completely transitive codes with minimum distance at least 9, showing that the *binary repetition code* (see Definition 2.5) is the unique code in this family.

In their paper [16], the second and third authors also generalised coset-completely transitive codes by introducing *completely transitive codes*, which are defined for not necessarily linear codes over an arbitrary alphabet, and are also completely regular. There exist completely transitive codes that are not coset-completely transitive. For example, the first and third authors proved that certain Hadamard codes [13] and the Nordstrom-Robinson codes [14] are completely transitive, but as they are non-linear, cannot

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be coset-completely transitive; also the repetition code of length 3 over a finite field \mathbb{F}_q for $q \geq 9$ is an example of a linear completely transitive code that is not coset-completely transitive. In this paper we begin the classification of this class of completely regular codes. To do this we consider codes as subsets of the vertex set of the Hamming graph $H(m, q)$, which is a natural setting to study codes of length m over a finite alphabet Q of size q . The automorphism group of $\Gamma = H(m, q)$, denoted by $\text{Aut}(\Gamma)$, is isomorphic to $S_q \text{wr} S_m$, and via the homomorphism given in (2.4), has an action on the entries of codewords with kernel $\mathfrak{B} \cong S_q^m$. Given any code C in $H(m, q)$ with covering radius ρ , we define the *distance partition* of C , $\{C_0 = C, C_1, \dots, C_\rho\}$, which is a partition of the vertex set of $H(m, q)$ (see Section 2.1). If there exists $X \leq \text{Aut}(\Gamma)$ such that C_i is an X -orbit for $i = 0, \dots, \rho$, we say C is *X -completely transitive*, or simply *completely transitive*, and we prove the following.

Theorem 1.1. *Let C be a code in $H(m, q)$ with minimum distance δ such that $|C| \geq 2$ and $\delta \geq 5$. Then C is X -completely transitive with $X \cap \mathfrak{B} = 1$ if and only if $q = 2$, C is equivalent to the binary repetition code, and $X \cong S_m$.*

In Section 2 we introduce the necessary definitions and preliminary results required for this paper. For the remainder of the paper we consider X -completely transitive codes with $X \cap \mathfrak{B} = 1$. In particular in Section 3, we consider such codes with $X \cong A_m$ or S_m , and with $\delta = m$. We deduce that for X -completely transitive codes with $\delta \geq 5$ and $X \cap \mathfrak{B} = 1$, the group X has a 2-transitive action on entries, and therefore is of affine or almost simple type, and in Section 4.1 and Section 4.2 we consider the respective cases. Finally in Section 5 we prove Theorem 1.1.

2. DEFINITIONS AND PRELIMINARIES

2.1. Codes in Hamming Graphs. The *Hamming graph* $H(m, q)$ is the graph Γ with vertex set $V(\Gamma)$, the set of m -tuples with entries from an *alphabet* Q of size q , and an edge exists between two vertices if and only if they differ in precisely one entry. Throughout we assume that $m, q \geq 2$. Any code of length m over an alphabet Q of size q can be embedded as a subset of $V(\Gamma)$. The automorphism group of Γ , which we denote by $\text{Aut}(\Gamma)$, is the semi-direct product $\mathfrak{B} \rtimes \mathfrak{L}$ where $\mathfrak{B} \cong S_q^m$ and $\mathfrak{L} \cong S_m$, see [3, Thm. 9.2.1]. Let $g = (g_1, \dots, g_m) \in \mathfrak{B}$, $\sigma \in \mathfrak{L}$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in V(\Gamma)$. Then $g\sigma$ acts on α in the following way:

$$(2.1) \quad \alpha^{g\sigma} = (\alpha_{1^{\sigma^{-1}}}^{g_1^{\sigma^{-1}}}, \dots, \alpha_{m^{\sigma^{-1}}}^{g_m^{\sigma^{-1}}})$$

Let $M = \{1, \dots, m\}$, and view M as the set of vertex entries of $H(m, q)$. Let 0 denote a distinguished element of the alphabet Q . For $\alpha \in V(\Gamma)$, the *support* of α is the set $\text{supp}(\alpha) = \{i \in M : \alpha_i \neq 0\}$. The *weight* of α is defined as $\text{wt}(\alpha) = |\text{supp}(\alpha)|$. For any $a \in Q \setminus \{0\}$ we use the notation $(a^k, 0^{m-k})$ to denote the vertex in $V(\Gamma)$ that has a in the first k entries, and 0 in the remaining entries, and if $k = 0$ we denote the vertex by $\mathbf{0}$.

Lemma 2.1. *Let $\alpha = \mathbf{0}$ and $x = (g_1, \dots, g_m)\sigma \in \text{Aut}(\Gamma)_\alpha$. Then $\text{supp}(\beta^x) = \text{supp}(\beta)^\sigma$ for all $\beta \in V(\Gamma)$.*

Proof. Since each g_i fixes 0 and σ permutes coordinates, the i th entry of β^x is non-zero if and only if the $i^{\sigma^{-1}}$ entry of β is non-zero. Thus the result follows. \square

For all pairs of vertices $\alpha, \beta \in V(\Gamma)$, the *Hamming distance* between α and β , denoted by $d(\alpha, \beta)$, is defined to be the number of entries in which the two vertices differ. This is equal to the length of the shortest path in the graph between α and β . We let $\Gamma_k(\alpha)$ denote the set of vertices in $V(\Gamma)$ that are at distance k from α .

Let C be a code in $H(m, q)$. The *minimum distance* δ of C is the smallest distance between distinct codewords of C . For any vertex $\gamma \in V(\Gamma)$, the *distance of γ from C* is equal to $d(\gamma, C) = \min\{d(\gamma, \beta) : \beta \in C\}$. The *covering radius* ρ of C is the maximum distance any vertex in $H(m, q)$ is from C . We let C_i denote the set of vertices that are distance i from C , and deduce, for $i \leq \lfloor (\delta - 1)/2 \rfloor$, that C_i is the disjoint union of $\Gamma_i(\alpha)$ as α varies over C . Furthermore, $\{C = C_0, C_1, \dots, C_\rho\}$ forms a partition of $V(\Gamma)$, called the *distance partition of C* . The *distance distribution of C* is the $(m + 1)$ -tuple $a(C) = (a_0, \dots, a_m)$ where

$$(2.2) \quad a_i = \frac{|\{(\alpha, \beta) \in C \times C : d(\alpha, \beta) = i\}|}{|C|}.$$

We observe that $a_i \geq 0$ for all i and $a_0 = 1$. Moreover, $a_i = 0$ for $1 \leq i \leq \delta - 1$ and $|C| = \sum_{i=0}^m a_i$. In the case where q is a prime power, the *MacWilliams transform* of $a(C)$ is the $(m + 1)$ -tuple $a'(C) = (a'_0, \dots, a'_m)$ where

$$(2.3) \quad a'_k := \sum_{i=0}^m a_i K_k(i)$$

with

$$K_k(x) := \sum_{j=0}^k (-1)^j \binom{x}{j} \binom{m-x}{k-j} (q-1)^{k-j}.$$

It follows from [19, Lem. 5.3.3] that $a'_k \geq 0$ for $k = 0, 1, \dots, m$.

For $\alpha = (\alpha_i)$, $\beta = (\beta_i) \in V(\Gamma)$, we let $\text{Diff}(\alpha, \beta) = \{i \in M : \alpha_i \neq \beta_i\}$. Now suppose $|C| \geq 2$ and $\alpha, \beta \in C$. Then we let

$$\text{Diff}(\alpha, \beta, C) = \{\gamma \in C : \text{Diff}(\alpha, \gamma) = \text{Diff}(\alpha, \beta)\}.$$

By definition, $\beta \in \text{Diff}(\alpha, \beta, C)$, so $\text{Diff}(\alpha, \beta, C) \neq \emptyset$.

Lemma 2.2. *Let C be a code with minimum distance δ and $|C| \geq 2$, and let $\alpha, \beta \in C$ such that $d(\alpha, \beta) = \delta$. Then for all $a \in Q$, there exists $x \in \text{Aut}(\Gamma)$ such that the following two conditions hold.*

- (i) $\alpha^x = (a, \dots, a)$, and
- (ii) for each $\gamma \in \text{Diff}(\alpha, \beta, C)$, $\gamma^x = (c^\delta, a^{m-\delta})$ for some $c \in Q \setminus \{a\}$.

Proof. Let $\text{Diff}(\alpha, \beta, C) = \{\beta^1, \dots, \beta^s\}$. It follows that $\beta^i|_k = \alpha_k$ for each $i \leq s$ and $k \in M \setminus \text{Diff}(\alpha, \beta)$. Therefore, because C has minimum distance δ , $d(\beta^i, \beta^j) = \delta$ for each distinct pair $\beta^i, \beta^j \in \text{Diff}(\alpha, \beta, C)$. This implies that for each $k \in \text{Diff}(\alpha, \beta)$, the $s+1$ entries $\alpha_k, \beta^1|_k, \dots, \beta^s|_k$ are pairwise distinct elements of Q . Thus $s \leq q-1$. Let $a \in Q$ and $\{c_1, \dots, c_s\} \subseteq Q \setminus \{a\}$. Since S_q acts q -transitively on Q , it follows

that for each $k \in \text{Diff}(\alpha, \beta)$ there exists $h_k \in S_q$ such that $(\beta^i|_k)^{h_k} = c_i$ for each $i \leq s$ and $\alpha_k^{h_k} = a$. Also for each $k \in M \setminus \text{Diff}(\alpha, \beta)$ let $h_k = (a \ \alpha_k) \in S_q$. Now let $h = (h_1, \dots, h_m) \in \mathfrak{B}$. Since S_m acts m -transitively on M and $|\text{Diff}(\alpha, \beta)| = \delta \leq m$, there exists $\sigma \in S_m$ such that $\text{Diff}(\alpha, \beta)^\sigma = \{1, \dots, \delta\}$. Let $x = h\sigma \in \text{Aut}(\Gamma)$. Then $\alpha^x = (a, \dots, a)$ and $(\beta^i)^x = (c_i^\delta, a^{m-\delta})$ for each $i = 1, \dots, s$. \square

We say two codes C and C' in $H(m, q)$ are *equivalent* if there exists $x \in \text{Aut}(\Gamma)$ such that $C^x = C'$. If $C = C'$, then x is an *automorphism* of C , and the *automorphism group* of C is the setwise stabiliser of C in $\text{Aut}(\Gamma)$, which we denote by $\text{Aut}(C)$.

Finally, for a set Ω and group $G \leq \text{Sym}(\Omega)$, we say G acts *k -homogeneously* on Ω if G acts transitively on $\Omega^{\{k\}}$, the set of k -subsets of Ω .

2.2. s -Neighbour transitive codes.

Definition 2.3. Let C be a code in $H(m, q)$ with distance partition $\{C, C_1, \dots, C_\rho\}$ and s be an integer with $0 \leq s \leq \rho$. If there exists $X \leq \text{Aut}(\Gamma)$ such that C_i is an X -orbit for $i = 0, \dots, s$, we say C is (X, s) -neighbour transitive, or simply s -neighbour transitive. We observe that (X, s) -neighbour transitive codes are necessarily (X, k) -neighbour transitive for all $k \leq s$. Moreover, X -completely transitive codes (defined in Section 1) correspond to (X, ρ) -neighbour transitive codes.

Remark 2.4. Let $y \in \text{Aut}(\Gamma)$, and let C be an (X, s) -neighbour transitive code with minimum distance δ . By following a similar argument to that used in [15, Sec. 2], it holds that C^y is (X^y, s) -neighbour transitive, and because minimum distance is preserved by equivalence, C^y has minimum distance δ . Thus for any $a \in Q \setminus \{0\}$, Lemma 2.2 allows us to replace C with an equivalent (X, s) -neighbour transitive code with minimum distance δ that contains $\mathbf{0}$ and $(a^\delta, 0^{(m-\delta)})$.

Let $X \leq \text{Aut}(\Gamma)$ and consider the following homomorphism:

$$(2.4) \quad \begin{array}{ccc} \mu : & X & \longrightarrow S_m \\ & g\sigma & \longmapsto \sigma \end{array}$$

Then μ defines an action of X on $M = \{1, \dots, m\}$, and the kernel of this action is equal to $X \cap \mathfrak{B}$. In this paper we are interested in X -completely transitive codes with $X \cap \mathfrak{B} = 1$, that is X has a faithful action on M . Hence, in this case we can identify X with $\mu(X)$.

Definition 2.5. The *repetition code* in $H(m, q)$, denoted by $\text{Rep}(m, q)$, is equal to the set of vertices of the form (a, \dots, a) , for all $a \in Q$. It has minimum distance $\delta = m$.

Example 2.6. Let $C = \text{Rep}(m, 2)$ and let α be the zero codeword. We show that C is X -completely transitive with $X \cong S_m$ as in Theorem 1.1. It is clear that $\mathfrak{L} \leq \text{Aut}(C)$ and that $H = \langle (h, \dots, h) \rangle \leq \text{Aut}(C)$, where $1 \neq h \in S_2$. In fact, $\text{Aut}(C) = \langle H, \mathfrak{L} \rangle \cong H \times \mathfrak{L}$ [15]. Now let X be the group consisting of automorphisms of the form $x = (h, \dots, h)\sigma$ if σ is an odd permutation and $x = \sigma$ if σ is an even permutation. Then $X \cong S_m$, $X_\alpha \cong A_m$, $X \cap \mathfrak{B} = 1$, and X acts transitively on C . The covering radius of C is $\lfloor \frac{m}{2} \rfloor$ and C_i consists of the vertices of weights i and $m-i$, for $i = 0, \dots, \lfloor \frac{m}{2} \rfloor$. Let $\nu_1, \nu_2 \in C_i$. If ν_1, ν_2 both have the same weight, then because A_m acts i -homogeneously on M for

all $i \leq m$ it follows that there exists $\sigma \in X_\alpha$ such that $\nu_1^\sigma = \nu_2$. Now suppose ν_1 and ν_2 have different weights, say ν_1 has weight i and ν_2 has weight $m-i$. Then there exists $x \in X$ such that ν_2^x has weight i . Consequently there exists $\sigma \in X_\alpha$ such that $\nu_1^\sigma = \nu_2^x$, thus $\nu_1^{\sigma x^{-1}} = \nu_2$. Hence X acts transitively on C_i and so C is X -completely transitive.

Proposition 2.7. *Let C be an (X, s) -neighbour transitive code with minimum distance δ . Then for $\alpha \in C$ and $i \leq \min(s, \lfloor \frac{\delta-1}{2} \rfloor)$, the stabiliser X_α fixes setwise and acts transitively on $\Gamma_i(\alpha)$. In particular, X_α acts i -homogeneously on M .*

Proof. By replacing C with an equivalent code if necessary, Remark 2.4 allows us to assume that $\alpha = \mathbf{0} \in C$. Firstly, because automorphisms of the Hamming graph preserve distance, it follows that $X_\alpha \leq X_{\Gamma_i(\alpha)}$. Now let $\nu_1, \nu_2 \in \Gamma_i(\alpha)$. As C_i is an X -orbit, and because $\Gamma_i(\alpha) \subseteq C_i$, there exists $x \in X$ such that $\nu_1^x = \nu_2$. Suppose $x \notin X_\alpha$. Then $\alpha \neq \alpha^x \in C$, and so $d(\alpha, \alpha^x) \geq \delta$. However, $d(\alpha, \alpha^x) \leq 2i < \delta$, which is a contradiction. Thus X_α acts transitively on $\Gamma_i(\alpha)$.

Finally, let $J_1, J_2 \in M^{\{i\}}$, and $\nu, \gamma \in V(\Gamma)$ such that $\text{supp}(\nu) = J_1$ and $\text{supp}(\gamma) = J_2$. It follows that $\nu, \gamma \in \Gamma_i(\alpha) \subseteq C_i$. As X_α acts transitively on $\Gamma_i(\alpha)$, there exists $x = (g_1, \dots, g_m)\sigma \in X_\alpha$ such that $\nu^x = \gamma$. A consequence of Lemma 2.1 is that $J_1^\sigma = \text{supp}(\nu)^\sigma = \text{supp}(\nu^x) = \text{supp}(\gamma) = J_2$. Hence X_α acts i -homogeneously on M . \square

Corollary 2.8. *Let C be an (X, s) -neighbour transitive code with minimum distance δ . Then for each $i \leq \min(s, \lfloor \frac{\delta-1}{2} \rfloor)$ and $I \in M^{\{i\}}$, the setwise stabiliser X_I acts transitively on C .*

Proof. By definition C is (X, i) -neighbour transitive and, by Proposition 2.7, X_α acts transitively on the set $M^{\{i\}}$ of i -subsets of M . Hence X is transitive on $C \times M^{\{i\}}$, and so X_I is transitive on C . \square

Let $X \leq \text{Aut}(\Gamma)$. Then for each $i \in M$ we define an action of $X_i = \{g\sigma \in X : i^\sigma = i\}$ on the alphabet Q via the following homomorphism:

$$\begin{aligned} \varphi_i : \quad X_i &\longrightarrow S_Q \\ (g_1, \dots, g_m)\sigma &\longmapsto g_i \end{aligned}$$

We denote the image of X_i under φ_i by X_i^Q .

Proposition 2.9. *Let C be an $(X, 1)$ -neighbour transitive code in $H(m, q)$ with $\delta \geq 3$ and $|C| > 1$. Then X_1^Q acts 2-transitively on Q .*

Proof. Let $a \in Q \setminus \{0\}$. By replacing C with an equivalent code if necessary, Remark 2.4 allows us to assume that $\alpha = \mathbf{0}$ and $\beta = (a^\delta, 0^{m-\delta})$ are two codewords of C . Choose any $b \in Q \setminus \{0\}$. As $\delta \geq 3$ it follows that $\nu_1 = (a, 0^{m-1}), \nu_2 = (b, 0^{m-1}) \in \Gamma_1(\alpha) \subseteq C_1$. By Proposition 2.7, there exists $x = (g_1, \dots, g_m)\sigma \in X_\alpha$ such that $\nu_1^x = \nu_2$. Consequently, Lemma 2.1 implies that $1^\sigma = 1$. Thus $a^{g_1} = b$, and because $x \in X_\alpha$, we conclude that $g_1 \in (X_1^Q)_0$, the stabiliser of 0 in X_1^Q . Hence $(X_1^Q)_0$ acts transitively on $Q \setminus \{0\}$. By Corollary 2.8, X_1 acts transitively on C . Hence there exists $y = (h_1, \dots, h_m)\pi \in X_1$ such that $\alpha^y = \beta$. As $y \in X_1$ we have that $0^{h_1} = a$ and $h_1 \in X_1^Q$. Thus X_1^Q acts 2-transitively on Q . \square

Corollary 2.10. *Let C be an $(X, 2)$ -neighbour transitive code with $|C| > 1$, $X \cap \mathfrak{B} = 1$ and $\delta \geq 5$. Then X acts 2-transitively on M .*

Proof. By Proposition 2.7, X_α acts 2-homogeneously on M , and so X has a faithful 2-homogeneous action on M . By Proposition 2.9, X_1^Q acts 2-transitively on Q . Thus X_1^Q has even order, and so X has even order. Therefore, by [24, Lem. 2.1], X acts 2-transitively on M . \square

Lemma 2.11. *Let C be an X -completely transitive code. Then $q^m/(m+1) \leq |X|$. Moreover, if $X \lesssim S_m$ and $m \geq 5$, then $q \leq m - 2$.*

Proof. Since the diameter of $H(m, q)$ is equal to m , we naturally have that $\rho \leq m$. Then, because $V(\Gamma)$ has size q^m and the distance partition of C has $\rho + 1$ parts, there exists i such that

$$|C_i| \geq \frac{q^m}{\rho + 1} \geq \frac{q^m}{m + 1}.$$

As C_i is an X -orbit it follows that $|C_i| \leq |X|$, and so the first inequality holds. Now suppose $X \lesssim S_m$. Then $|X| \leq m!$ and so $q^m \leq (m + 1)!$. If $q \geq m - 1$ then $(m - 1)^m \leq (m + 1)!$, which holds if and only if $m \leq 4$. \square

2.3. s -regular and completely regular codes.

Definition 2.12. Let C be a code with covering radius ρ and s be an integer such that $0 \leq s \leq \rho$. We say C is s -regular if for each vertex $\gamma \in C_i$, with $i \in \{0, \dots, s\}$, and each integer $k \in \{0, \dots, m\}$, the number $|\Gamma_k(\gamma) \cap C|$ depends only on i and k . If $s = \rho$ we say C is *completely regular*.

It follows from the definitions that any code equivalent to an s -regular code is necessarily s -regular. The next three results examine the natural expectation that a completely regular code with large minimum distance would be small in size.

Lemma 2.13. *Let C be a code with $|C| \geq 2$ and $\delta = m$. Then there exists C' equivalent to C with $C' \subseteq \text{Rep}(m, q)$. Moreover if C is 1-regular then $C' = \text{Rep}(m, q)$; if C is 2-regular and $m \geq 5$ then $C' = \text{Rep}(m, 2)$.*

Proof. Let $0, a \in Q$. By Lemma 2.2, there is a code C' equivalent to C which contains $\alpha = (0, \dots, 0)$ and $\beta = (a, \dots, a)$, and each $\gamma \in C' \setminus \{\alpha, \beta\}$ at distance $\delta = m$ from α is of the form (b, \dots, b) for some $b \in Q \setminus \{0, a\}$. As C has $\delta = m$, it follows that C' is a subset of the repetition code $\text{Rep}(m, q)$. We note this implies $|C| = |C'| \leq q$.

Assume C , and hence C' , is 1-regular. Suppose $|C'| < q$. Then there exists $b \in Q \setminus \{0, a\}$ such that b does not appear in any codeword of C' . Let $\nu_1 = (a, 0, \dots, 0)$ and $\nu_2 = (b, 0, \dots, 0)$. Then $\nu_1, \nu_2 \in C'_1$. It follows that $|\Gamma_{m-1}(\nu_1) \cap C'| = 2$ if $m = 2$, and 1 if $m \geq 3$, while $|\Gamma_{m-1}(\nu_2) \cap C'| = 1$ if $m = 2$ and 0 if $m \geq 3$, which is a contradiction. Therefore $|C'| = q$ and $C' = \text{Rep}(m, q)$.

Now assume that $m \geq 5$ and C , and hence C' , is 2-regular. Let $\nu_3 = (a, a, 0^{m-2})$. As ν_3 has weight 2 and $m \geq 5$, we have that $\nu_3 \in C'_2$. Also, $d(\nu_3, \beta) = m - 2$. Therefore, because C' is 2-regular,

$\Gamma_{m-2}(\nu) \cap C' \neq \emptyset$ for all $\nu \in C'_2$. Now suppose that $q \geq 3$ and $b \in Q \setminus \{0, a\}$. Then $\nu_4 = (b, a, 0^{m-2}) \in C'_2$. Let $\gamma = (c, \dots, c)$, an arbitrary element of C' . Then

$$d(\nu_4, \gamma) = \begin{cases} 2 & \text{if } c = 0, \\ m-1 & \text{if } c = a \text{ or } b, \\ m & \text{if } c \in Q \setminus \{0, a, b\}. \end{cases}$$

Since $m \geq 5$ it follows that $\Gamma_{m-2}(\nu_4) \cap C = \emptyset$ which is a contradiction. Therefore $q = 2$. \square

Lemma 2.14. *Let C be a 1-regular code in $H(m, q)$ with $|C| = 2$. Then $\delta = 1$ or m , and $q = 2$. Moreover, if $\delta = m$ then C is equivalent to the binary repetition code $\text{Rep}(m, 2)$.*

Proof. Firstly suppose that $\delta < m$. By Lemma 2.2, C is equivalent to

$$C' = \{(0, \dots, 0), (a^\delta, 0^{m-\delta})\}$$

for some $a \in Q \setminus \{0\}$, and C' is 1-regular since C is. Suppose $\delta \geq 2$, and let $\nu_1 = (a, 0^{m-1})$ and $\nu_2 = (0^\delta, a, 0^{m-\delta-1})$. Since $2 \leq \delta < m$ it follows that $\nu_1, \nu_2 \in C'_1$. However, we observe that if $\delta \geq 3$ then $|\Gamma_{\delta-1}(\nu_1) \cap C'| = 1$ and $|\Gamma_{\delta-1}(\nu_2) \cap C'| = 0$, while if $\delta = 2$ then $|\Gamma_1(\nu_1) \cap C'| = 2$ and $|\Gamma_1(\nu_2) \cap C'| = 1$, contradicting the fact that C' is 1-regular. Thus $\delta = 1$. Now suppose that $q \geq 3$ and $b \in Q \setminus \{0, a\}$. Let $\nu_3 = (b, 0^{m-1})$ and $\nu_4 = (0, b, 0^{m-2})$. Then $\nu_3, \nu_4 \in C'_1$. However, $|\Gamma_1(\nu_3) \cap C'| = 2$ and $|\Gamma_1(\nu_4) \cap C'| = 1$, contradicting the fact that C' is 1-regular. Thus $q = 2$. Now suppose that $\delta = m$. Then Lemma 2.13 implies that C is equivalent to the repetition code $\text{Rep}(m, q)$. Since $|\text{Rep}(m, q)| = q$ and $|C| = 2$, it follows that $q = 2$. \square

Lemma 2.15. *Let C be a completely regular code in $H(m, q)$ with $m \geq 5$ and $\delta \geq 2$. Then $|C| = 2$ if and only if $\delta = m$.*

Proof. Suppose that $|C| = 2$. Since $m \geq 5$ it follows that C is 1-regular. As $\delta \geq 2$, Lemma 2.14 implies that $\delta = m$. Conversely suppose that $\delta = m$. As C is completely regular, $m \geq 5$ and $\delta = m$ it follows that $\rho \geq 2$, and so C is 2-regular. Therefore Lemma 2.13 implies that C is equivalent to $\text{Rep}(m, 2)$. Thus $|C| = 2$. \square

2.4. t-designs and q-ary t-designs. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ where \mathcal{P} is a set of *points* of cardinality m , and \mathcal{B} is a set of k -subsets of \mathcal{P} called *blocks*. We say \mathcal{D} is a $t - (m, k, \lambda)$ *design* if every t -subset of \mathcal{P} is contained in exactly λ blocks of \mathcal{B} . We let b denote the number of blocks in \mathcal{D} and r denote the number of blocks that contain any given point. We say a non-negative integer ℓ is *block intersection number* of \mathcal{D} if there exist distinct blocks $B, B' \in \mathcal{B}$ such that $|B \cap B'| = \ell$. An *automorphism* of \mathcal{D} is a permutation of \mathcal{P} that preserves \mathcal{B} , and we let $\text{Aut}(\mathcal{D})$ denote the group of automorphisms of \mathcal{D} . For further concepts and definitions about t -designs, see [6].

Remark 2.16. Let \mathcal{P} be a set with cardinality m and $G \leq \text{Sym}(\mathcal{P})$. Suppose G acts t -homogeneously on \mathcal{P} , and let $B \in \mathcal{P}^{\{k\}}$. Then (\mathcal{P}, B^G) forms a $t - (m, k, \lambda)$ design for some integer λ . Using this fact, we can prove that $\text{PSL}(2, 5)$ has two orbits, $\mathcal{O}_1, \mathcal{O}_2$, on $M^{\{3\}}$ (here $M = \{1, \dots, 6\}$), each of which is a $2 - (6, 3, 2)$ design, and each is the complementary design of the other (see [11, Sec. 2.4 and

Lem. 9.1.1]). Also, any design with these parameters is unique up to isomorphism and has automorphism group isomorphic to $\text{PSL}(2, 5)$.

For $\alpha, \beta \in V(\Gamma)$, we say α is *covered* by β if $\alpha_i = \beta_i$ for each non-zero component α_i of α . Let \mathcal{D} be a non-empty set of vertices of weight k in $H(m, q)$. Then we say \mathcal{D} is a q -ary t -(m, k, λ) *design* if for every vertex ν of weight t , there exist exactly λ vertices of \mathcal{D} that cover ν . If $q = 2$, this definition coincides with the usual definition of a t -design, in the sense that the set of blocks of the t -design is the set of supports of vertices in \mathcal{D} , and as such we simply refer to 2-ary t -designs as t -designs. It is known that for a completely regular code C in $H(m, q)$ with zero codeword and minimum distance δ , the set $C(k)$ of codewords of weight k forms a q -ary t -(m, k, λ) design for some λ with $t = \lfloor \frac{\delta}{2} \rfloor$ [17]. Using this, we prove the following results.

Lemma 2.17. *Let C be a completely regular code in $H(m, 2)$ with $|C| \geq 2$ and $5 \leq \delta < m$. Then $|C| \geq m + 1$.*

Proof. C is equivalent to a completely regular code C' that contains $\mathbf{0}$. As $\delta \geq 5$, it follows that $C'(\delta)$ is a 2-(m, δ, λ_2) design for some λ_2 [6, Cor. 1.6]. Since $\delta < m$, Fisher's inequality [6, Thm. 1.14] implies that $|C'(\delta)| \geq m$. Consequently $|C| = |C'| \geq m + 1$. \square

Lemma 2.18. *There do not exist binary completely regular codes of length m with minimum distance δ for $m = 13$ and $\delta = 5, 6$, or for $m = 16$ and $\delta = 5, 7, 8$.*

Proof. Let C be a binary completely regular code of length 16 with $\delta = 5$. By replacing C with an equivalent code if necessary, we can assume that $\mathbf{0} \in C$. Therefore $C(5)$ forms a 2-(16, 5, λ) design for some λ . It follows that $r = 15\lambda/4$, and so 4 divides λ . There are exactly λ codewords of weight 5 whose support contains $\{1, 2\}$. Because $\delta = 5$, it follows that the supports of any pair of these codewords intersect precisely in $\{1, 2\}$. Consequently

$$\lambda \leq \frac{16 - 2}{5 - 2} < 5,$$

and so $\lambda = 4$. Thus $C(5)$ forms a 2-(16, 5, 4) design and $a_5 = |C(5)| = 48$. Using the fact that $\delta = 5$, a simple counting argument gives that $C(5)$ has block intersection numbers 2, 1 and 0. Consequently, for $\alpha \in C(5)$, it holds that $\Gamma_k(\alpha) \cap C(5) \neq \emptyset$ for $k = 6, 8, 10$, and so $C(k) \neq \emptyset$ for the same values of k .

Suppose that the all one vertex $\mathbf{1}$ is not a codeword. Then, by [12, Lemma 2.2], C has covering radius $\rho \geq \delta - 1 = 4$ and $C_\rho = \mathbf{1} + C$. Furthermore, because $C(10) \neq \emptyset$ and $\mathbf{1} \in C_\rho$, it follows that $\rho \leq 6$. It is known that C_ρ is also completely regular with distance partition $\{C_\rho, C_{\rho-1}, \dots, C_1, C\}$ [21]. Thus, as $\delta = 5$, it follows that $C_{\rho-i} = \mathbf{1} + C_i$ for $i = 1, 2$ also. Therefore if $\rho = 4$ or 5 it holds that $|C|(2 + 2 \times 16 + \binom{16}{2}) = 2^{16}$ or $|C|(2 + 2 \times 16 + 2 \times \binom{16}{2}) = 2^{16}$ respectively, which is a contradiction. Hence $\rho = 6$. As $\mathbf{1} \in C_6$, it follows that $k = 10$ is the maximum weight of any codeword in C . Thus the distance distribution of C is equal to

$$a(C) = (1, 0, 0, 0, 0, 48, a_6, a_7, a_8, a_9, a_{10}, 0, 0, 0, 0, 0, 0)$$

By considering the MacWilliams transform of $a(C)$ (see (2.3)), we obtain the following linear constraints [19, Lem. 5.3.3]:

$$\begin{aligned} 600 - 6a_7 - 8a_8 - 6a_9 &\geq 0 \\ -360 + 6a_7 - 8a_8 + 6a_9 &\geq 0 \end{aligned}$$

with $a_7, a_9 \geq 0$ and $a_8 > 0$. Adding these together implies that $a_8 \leq 15$. However, there exists a positive integer λ' such that $C(8)$ forms a $2 - (16, 8, \lambda')$ design with

$$a_8 = |C(8)| = \frac{16 \cdot 15}{8 \cdot 7} \lambda' = \frac{30}{7} \lambda'.$$

Thus 7 divides λ' and $a_8 \geq 30$, which is a contradiction. Hence $\mathbf{1} \in C$. This implies that C is *antipodal*, that is, $\alpha + \mathbf{1} \in C$ for all $\alpha \in C$, and so $a_i = a_{m-i}$ for all i in $a(C)$. Again, by applying the MacWilliams transform to $a(C)$ we generate twelve linear constraints that must be non-negative. However, it is straight forward to obtain a contradiction from these constraints (see [11, Lem. 7.4.2.2]). Thus no such code exists with $m = 16$ and $\delta = 5$. For the other values of m , δ , we follow a similar argument to that given in [1, Lem. 6] to prove that binary completely regular codes with these parameters do not exist (see [11, Lem. 7.4.2.1]). \square

3. BASIC CASES

We now begin to prove Theorem 1.1. We first consider the case $X \cong A_m$ or S_m , and then the case $\delta = m$.

Remark 3.1. If C is an X -completely transitive code, then C is completely regular [16]. Furthermore, if $\delta \geq 5$ then C has covering radius $\rho \geq 2$. Thus C is at least $(X, 2)$ -neighbour transitive, so by Proposition 2.7, X_α acts 2-homogeneously on M . As we only consider completely transitive codes with $\delta \geq 5$ for the remainder of this paper, from now on we use both these results without further reference.

Proposition 3.2. *Let C be an X -completely transitive code in $H(m, q)$ with $|C| > 1$, $X \cap \mathfrak{B} = 1$, $X \cong A_m$ or S_m and $\delta \geq 5$. Then $q = 2$, $X \cong S_m$, $X_\alpha \cong A_m$ and C is equivalent to $\text{Rep}(m, 2)$.*

Proof. As $m \geq \delta \geq 5$ the code C is at least 2-regular. If $m = 5$ then by Lemma 2.13, $q = 2$ and C is equivalent to $\text{Rep}(m, 2)$. In this case, since X is transitive on C , X_α has index 2 and hence is normal in X . Thus $X \cong S_5$ and $X_\alpha \cong A_5$. Thus we may assume that $m \geq 6$. Since $X \cong A_m$ or S_m , it follows that (the stabiliser of the first entry) $X_1 \cong A_{m-1}$ or S_{m-1} . By Proposition 2.9, X_1 has a 2-transitive action of degree q . By Lemma 2.11, for $m \geq \delta \geq 5$, we have that $q \leq m - 2$. Thus, by considering the 2-transitive actions of A_n and S_n for an arbitrary n [4], we have, since $m \geq 6$, that $X \cong S_m$ and $q = 2$.

Now consider the group X_α , and suppose first that A_m is not a subgroup of X_α . As $q = 2$ it follows that $|X : X_\alpha| = |C| \leq 2^m$, and since X_α acts 2-homogeneously and hence primitively on M , a result by Maróti [20] gives us that $|X_\alpha| \leq 3^m$. It follows that $m!/2^m = |X|/2^m \leq |X_\alpha| \leq 3^m$. Thus $m! \leq 6^m$, which implies that $m \leq 13$. By the Sphere Packing Bound [19, Thm. 5.2.7], $|C|(1 + m + \binom{m}{2}) \leq 2^m$, and so $|X_\alpha| \geq m!(1 + m + \binom{m}{2})/2^m$. Now, from [5] and [18], the only 2-homogeneous groups with

degree $m \leq 13$ that are not A_m or S_m are the projective groups, the affine groups, M_{11} with degree 11 or 12, and M_{12} with degree 12. However, we see in each case that the orders of these groups are always less than $m!(1 + m + \binom{m}{2})/2^m$, which is a contradiction. Thus A_m is a subgroup of X_α . Since $|C| > 1$, it follows that $X_\alpha \cong A_m$ and $|C| = 2$. Therefore, by Lemmas 2.14 and 2.15, C is equivalent to $\text{Rep}(m, 2)$. \square

Proposition 3.3. *Let C be an X -completely transitive code with $m \geq 5$, $|C| \geq 2$, $X \cap \mathfrak{B} = 1$ and $\delta = m$. Then C is equivalent to the repetition code $\text{Rep}(m, 2)$, $X \cong S_m$ and $X_\alpha \cong A_m$.*

Proof. As C is completely regular with $\delta = m \geq 5$, it follows that $\rho \geq 2$ and so C is at least 2-regular. Thus, by Lemma 2.13, C is equivalent to the repetition code $\text{Rep}(m, 2)$, so we just need to prove the statement about the groups X and X_α . By replacing C with an equivalent code if necessary, let us assume that $C = \text{Rep}(m, 2)$. As $|C| = 2$ we have that $|X : X_\alpha| = 2$. Furthermore, by Corollary 2.8, X_1 acts transitively on C , and so $|X_1 : X_{1,\alpha}| = 2$. We claim that $A_m \lesssim X$, from which, by Proposition 3.2, we obtain $X \cong S_m$ and $X_\alpha \cong A_m$. We repeatedly use the classification of 2-transitive groups to prove this claim (see [5]).

Suppose to the contrary that $A_m \not\lesssim X$. By Proposition 2.7, X_α (and so X also) is i -homogeneous on M for all $i \leq \lfloor \frac{\delta-1}{2} \rfloor = \lfloor \frac{m-1}{2} \rfloor$, and note that any i -homogeneous group is also $(m-i)$ -homogeneous. By the classification of 2-transitive groups, X is not 6-transitive (see [10, Sec. 7.3]), and hence is not 6-homogeneous by [18]. Thus $m \leq 12$ and if m is odd then $m = 5$, or $m = 9$ with $\text{PGL}(2, 8) \leq X_\alpha < X \leq \text{P}\Gamma\text{L}(2, 8)$. However in the latter case $|X : X_\alpha| = 1$ or 3, which is a contradiction. Also, if $m = 5$ then, by [10, Thm. 9.4B], $X \leq Z_5.Z_4$, since $A_5 \not\lesssim X$, and so $X_\alpha \lesssim D_{10}$, which is not 2-homogeneous, a contradiction. Thus $m \in \{6, 8, 10, 12\}$ and X, X_α are $(\frac{m-2}{2})$ -homogeneous on M .

If $m = 12$ then X, X_α are 5-transitive by [18], and the only possibility is $X \cong M_{12}$, which has no index 2-subgroup X_α . Similarly, if $m = 10$ then X, X_α are 4-transitive by [18], but the only 4-transitive subgroups of S_{10} are A_{10} and S_{10} . Next suppose $m = 8$. In this case $C = \text{Rep}(m, 8)$, which has covering radius $\rho = 4$. The only 3-homogeneous subgroup X of S_8 , not containing A_8 , with a subgroup of index 2 is $X \cong \text{PGL}(2, 7)$, with $X_\alpha \cong \text{PSL}(2, 7)$. However, since C is X -completely transitive, X is transitive on C_4 , the set of $\binom{8}{4} = 70$ vertices of weight 4. This is impossible since $|X|$ is not divisible by 5. Thus $m = 6$.

In this final case, $C = \text{Rep}(2, 6)$, which has covering radius $\rho = 3$, and C_3 consists of the 20 weight 3 vertices in $H(6, 2)$. The only 2-homogeneous subgroup X of S_6 , not containing A_6 , with an index 2 subgroup is $X \cong \text{PGL}(2, 5)$, with $X_\alpha \cong \text{PSL}(2, 5)$. We note that because $q = 2$, it follows that $X_\alpha = X \cap \mathfrak{L} \leq \mathfrak{L}$, where $\alpha = \mathbf{0}$. Let $H = N_{\mathfrak{L}}(X_\alpha) \cong \text{PGL}(2, 5)$. Note also that if $g = (h, \dots, h) \in \mathfrak{B}$ for $1 \neq h \in S_2$, then $X \leq \text{Aut}(C) = \langle g, \mathfrak{L} \rangle$. Suppose that $x = g\sigma \in X$ with $\sigma \in X_\alpha$. Then $x\sigma^{-1} = g \in X$, and so $X \cap \mathfrak{B}$ is a non-trivial normal 2-subgroup. However this contradicts the fact $X \cong \text{PGL}(2, 5)$. Therefore we deduce that $X = X_\alpha \cup g(H \setminus X_\alpha)$. By Remark 2.16, the induced action of X_α on $M^{\{3\}}$, the set of 3-subsets of M , has two orbits, $\mathcal{O}_1, \mathcal{O}_2$. Moreover, each orbit forms a $2 - (6, 3, 2)$ design and is the complementary design of the other. Also, because X acts transitively in its induced action on

$M^{\{3\}}$, and because $\text{PSL}(2, 5) \leq \text{PGL}(2, 5)$ we have that $\Delta = \{\mathcal{O}_1, \mathcal{O}_2\}$ is a system of imprimitivity for the action of X on $M^{\{3\}}$. Let $C(\mathcal{O}_i)$ be the set of vertices in $H(6, 2)$ whose supports are the elements of \mathcal{O}_i for each i , so $C_3 = C(\mathcal{O}_1) \cup C(\mathcal{O}_2)$. If $x \in X_\alpha$ it follows that $C(\mathcal{O}_1)^x = C(\mathcal{O}_1)$. If $x \in X \setminus X_\alpha$ then $x = g\sigma$ with $\sigma \in H \setminus X_\alpha$. It follows that $C(\mathcal{O}_2)^\sigma = C(\mathcal{O}_1)$, and because \mathcal{O}_2 is the complementary design of \mathcal{O}_1 , $C(\mathcal{O}_1)^g = C(\mathcal{O}_2)$. Thus $C(\mathcal{O}_1)^x = C(\mathcal{O}_1)$. Consequently, C_3 is not an X -orbit, which is a contradiction. Thus the claim is proved. \square

4. NEW HYPOTHESIS

By Lemma 2.15, if C is completely regular in $H(m, q)$ with $m \geq 5$ and $\delta \geq 2$ then $|C| = 2$ if and only if $\delta = m$. Therefore, given Propositions 3.2 and 3.3, and Corollary 2.10, to complete the proof of Theorem 1.1, we only need to consider X -completely transitive codes with $\delta < m$ (which is equivalent to $|C| > 2$) such that $X \cap \mathfrak{B} = 1$, and X is a 2-transitive subgroup of S_m not containing A_m . We bring this together in the following hypothesis.

Hypothesis 4.1. Let C be an X -completely transitive code in $H(m, q)$ with $|C| > 2$, minimum distance δ satisfying $5 \leq \delta < m$, and $X \cap \mathfrak{B} = 1$ such that $\mu(X) \cong X$ is 2-transitive not containing A_m .

Lemma 4.2. Let C be an X -completely transitive code that satisfies Hypothesis 4.1. Then either (i) $q = 2$, $X_1^Q = S_2$ and $\delta \leq m/2$, or (ii) $q = 3$, $X_1^Q = S_3$ and $8 \leq m \leq 24$. Moreover, X_1 is not perfect.

Proof. Since X acts 2-transitively, it acts primitively on M , and because it does not contain A_m we have that $|X| \leq 3^m$ for $m \leq 24$, and $|X| \leq 2^m$ otherwise [20]. By Lemma 2.11, $q^m/(m+1) \leq |X|$ from which we deduce that either $q = 2$; $q = 3$ and $m \leq 24$; or $q = 4$ and $m \leq 7$. The only binary completely regular code with $m/2 < \delta < m$ has minimum distance 4 [12]. Therefore, because $\delta \geq 5$, if $q = 2$ it follows that $\delta \leq m/2$, which also implies that $m \geq 10$. Suppose now that $q \in \{3, 4\}$. If $m = 7$, then the only 2-transitive groups X (not containing A_7) are $X \cong \text{PSL}(3, 2)$ and $\text{AGL}(1, 7)$, so $|X| \leq 168 < 3^7/8$, a contradiction. If $m = 6$ then $X \cong \text{PSL}(2, 5)$ or $\text{PGL}(2, 5)$, so $q^6/7 \leq |X| \leq 120$, which implies that $q = 3$ and $X = \text{PSL}(2, 5)$. However this implies that $X_1 \cong D_{10}$, which does not act as S_3 on Q , contradicting Proposition 2.9. Since $m \geq 6$, we deduce that $q = 3$ and $8 \leq m \leq 24$. The claims about X_1^Q follow from Proposition 2.9. It follows that X_1^Q is soluble and, in particular, X_1 is not perfect. \square

4.1. X is 2-transitive of Affine Type. Let C be a code that satisfies Hypothesis 4.1. The group X acts faithfully and 2-transitively on M and so X is either of affine or almost simple type. We consider the affine case first.

Proposition 4.3. There are no X -completely transitive codes in $H(m, q)$ satisfying Hypothesis 4.1 such that X is of affine type.

Proof. Throughout this proof we repeatedly use the classification of 2-transitive groups (see [5]). Suppose C is an X -completely transitive code satisfying Hypothesis 4.1 such that X is of affine type. Then $X = NX_1 \lesssim \text{AGL}(n, r)$ for some n, r with r a prime and $m = r^n$, and with N the unique minimal

Row	r	n	q
1	2	3	3
2	2	4 or 5	2

TABLE 1. Possible r, n, q in Affine case

normal subgroup of X of order r^n . Recall also, by Lemma 4.2, that either $q = 2$ and $\delta \leq m/2$ (so $m \geq 10$), or $q = 3$ and $8 \leq m \leq 24$. We deduce from Lemma 2.11 that

$$(4.1) \quad q^{r^n} \leq |X|(r^n + 1) \leq |\text{AGL}(n, r)|(r^n + 1) \leq r^{n^2+2n},$$

and so

$$(4.2) \quad f_n(r) := \frac{r^n}{\log(r)} \leq \frac{n^2 + 2n}{\log(q)}.$$

We claim that r, n, q are as in one of the rows in Table 1. Suppose first that $r = 2$. If $q = 3$ then (4.2) implies that $2^n \leq (n^2 + 2n)(\log(2)/\log(3))$, and so $n \leq 3$. Furthermore, because $m = r^n \in [8, 24]$ when $q = 3$ it follows that $n = 3$ as in row 1. If $q = 2$ then (4.2) implies that $10 \leq 2^n < n^2 + 2n$, and so $n = 4$ or 5 as in row 2. Suppose now that $r \geq 3$. In this case, $f_n(r)$ is an increasing function for a fixed n . Thus (4.2) implies that

$$(4.3) \quad 3^n \leq \frac{(n^2 + 2n)\log(3)}{\log(q)}$$

If $q = 3$ we deduce that $n = 1$. Hence $f_1(r) \leq 3/\log(3)$, and so $r = 3$ and $m = 3$, which is a contradiction. Thus $q = 2$ with $m \geq 10$, and (4.3) implies that $n \leq 2$. If $n = 2$ then $f_2(r) \leq 8/\log(2)$, which holds only if $r = 3$ (recall r is a prime), and so $m = 9$, contradicting the fact $m \geq 10$. Thus $n = 1$. Consequently $f_1(r) \leq 3/\log(2)$, which holds only if $m = r \leq 9$, again a contradiction. Thus the claim holds.

Consider row 1, so X is a 2-transitive subgroup of $\text{AGL}(3, 2)$, and by Proposition 2.9, $|X_1| = |X \cap \text{GL}(3, 2)|$ is even. It follows that $X \cong \text{AGL}(3, 2)$, but then $X_1 \cong \text{GL}(3, 2)$ is perfect, contradicting Lemma 4.2. In row 2, $m = 16$ or 32 . Suppose that $m = 32$. Then $X \lesssim \text{AGL}(5, 2)$, and as before $|X_1|$ is even. This means that $X \not\lesssim \text{GL}(1, 32)$ (of order 31.5), and hence $X_1 \cong \text{GL}(5, 2)$. However, in this case X_1 is perfect contradicting Lemma 4.2. Thus $m = 16$ and $X_1 \lesssim \text{GL}(4, 2)$. By Lemma 4.2, $\delta \leq 8$, and by Lemma 2.18 there do not exist binary completely regular codes of length 16 with $\delta = 5, 7$ or 8 . Thus $\delta = 6$. Any completely regular code in $H(16, 2)$ with $\delta = 6$ is equivalent to the Nordstrom-Robinson code [14], which consists of 256 codewords. Thus $|C| = 256$. Furthermore, by Corollary 2.8, X_1 acts transitively on C , and so 256 divides $|\text{GL}(4, 2)|$, which is a contradiction. \square

4.2. X is 2-transitive of Almost Simple Type. In this section we consider codes that satisfy Hypothesis 4.1 such that X is of almost simple type. The group $\text{Aut}(\Gamma)$ has a natural action on $\Omega = Q \times M$ where $h\sigma \in \text{Aut}(\Gamma)$ maps (a, i) to (a^{h_i}, i^σ) . It is a consequence of Proposition 2.9, and the fact that X induces the 2-transitive group $\mu(X)$ on M , that X acts transitively on Ω . In this action, $\mathcal{B} = \{Q \times \{i\} : i \in M\}$ is a system of imprimitivity and $X_B = X_1$ where $B = Q \times \{1\}$, so X_1^Q is permutationally isomorphic to X_B^B . Furthermore, it is a consequence of a result by the third author with Schneider that there exists

Line	X	m	Conditions
1	$\text{PFL}(2, 8)$	28	
2	HS	176	
3	Co_3	276	
4	M_{11}	11	
5	$M_{22} \rtimes C_2$	22	
6	$\text{Sp}(2\ell, 2)$	$2^{2\ell-1} - 2^{\ell-1}$	$\ell \geq 3$
7	$\text{Sp}(2\ell, 2)'$	$2^{2\ell-1} + 2^{\ell-1}$	$\ell \geq 3$
8	$\supseteq \text{Ree}(r)$	$r^3 + 1$	$r = 3^f, f \geq 3$ and odd
9	$\supseteq \text{PSU}(3, r)$	$r^3 + 1$	$r \geq 3$
10	$\supseteq \text{PSL}(n, r)$	$\frac{r^n - 1}{r - 1}$	$n \geq 2$ and $(n, r) \neq (2, 2), (2, 3)$

TABLE 2. Possible X and m in Almost Simple Case

$g \in \text{Aut}(\Gamma)$ such that $X^g \leq X_1^Q \text{ wr } \mu(X)$ [22]. The group X^g is of almost simple type, satisfies Hypothesis 1 of [9], and is faithful on \mathcal{B} . All groups with these properties are classified in [9, Thm. 1.4], and so the possibilities for X , m , $q = |B|$ are listed in [9, Tables 2 and 3]. However, recall from Lemma 4.2 that either $q = 3$ and $8 \leq m \leq 24$, or $q = 2$. The only possibilities in [9, Tables 2 and 3] that have $q = 3$ and $8 \leq m \leq 24$ are $\text{PSL}(n, r) \lesssim X \lesssim \text{PFL}(n, r)$ with $m = (r^n - 1)/(r - 1)$ for $(n, r) = (2, 16), (3, 3)$ or $(3, 4)$. In each case $3^m/(m + 1) > |X|$, contradicting Lemma 2.11. Thus $q = 2$, and the cases for which this holds in [9, Tables 2 and 3], excluding the Symmetric group case, are as in Table 2.

Proposition 4.4. *There are no X -completely transitive codes in $H(m, q)$ satisfying Hypothesis 4.1 such that X is of almost simple type.*

Proof. Throughout this proof, C is an X -completely transitive code in $H(m, q)$ that satisfies Hypothesis 4.1 such that X is of almost simple type. From our discussion above, $q = 2$ and X , m are as in one of the lines of Table 2. Moreover, by Lemma 4.2, $\delta \leq m/2$ and $m \geq 10$. We now consider each of the lines of Table 2, repeatedly using the classification of 2-transitive groups (see [5]).

Lines 1 – 3: In each case, $2^m/(m + 1) > |X|$, contradicting Lemma 2.11, and so no such code exists.

Line 4: In this case $X \cong M_{11}$ and $m = 11$. As $\delta \leq m/2$ it follows that $\delta = 5$. By the main result of [13], C is equivalent to the punctured Hadamard 12 code, and so $|C| = 24$. As X acts transitively on C we have that X_α is a subgroup of index 24 in M_{11} , and hence X_α is a subgroup of index 2 in a maximal subgroup isomorphic to $\text{PSL}(2, 11)$ (see [7]). However this contradicts the fact that $\text{PSL}(2, 11)$ is simple.

Line 5: In this case $X \cong M_{22} \rtimes C_2$, $m = 22$, X_α is 2-homogeneous of degree 22 and therefore 2-transitive [18]. However, the only 2-transitive proper subgroup of X is M_{22} , so $|C| \leq 2$, which is a contradiction.

Column	1	2	3
r	2	3 or 4	≤ 16
n	4	3	2

TABLE 3. Possible r, n in $\text{PSL}(n, r)$ case

Lines 6 – 7: In this case $m = 2^{2\ell-1} \pm 2^{\ell-1}$ with $\ell \geq 3$ and $|X| < 2^{(\ell^2+\ell)/2}$ [23, Table 4]. However, for $\ell \geq 3$, it holds that $m + 1 \leq 2^{2\ell-1} + 2^{\ell-1} + 1 < 2^{2\ell}$ and

$$m - 2\ell \geq 2^{2\ell-1} - 2^{\ell-1} - 2\ell \geq 2^{2\ell-2} \geq \frac{\ell^2 + \ell}{2}.$$

By Lemma 2.11, $|X| \geq 2^m/(m+1) > 2^{m-2\ell} \geq 2^{(\ell^2+\ell)/2}$, which is a contradiction.

Lines 8 – 9: Here $T \leq X \leq \text{Aut}(T)$ with $T \cong \text{PSU}(3, r)$ or $\text{Ree}(r)$, and $r = p^f \geq 3$ for a prime p and positive integer f . In both cases $|X| \leq (r^3 + 1)r^3(r^2 - 1)f \leq 2r^{12}/(r^3 + 2)$. By Lemma 2.11, $|X| \geq 2^{r^3+1}/(r^3 + 2)$ and hence $r^3 \log(2) \leq 12 \log(r)$. The expression $x^3/\log(x)$ is an increasing function in x for $x \geq e^{\frac{1}{3}}$. As $3^3/\log(3) > 12/\log(2)$ it follows that $r^3/\log(r) > 12/\log(2)$, which is a contradiction.

Lines 10: Here $\text{PSL}(n, r) \lesssim X \lesssim \text{P}\Gamma\text{L}(n, r)$ with $r = p^f$ for a prime p and $m = (r^n - 1)/(r - 1) < r^n$. By applying Lemma 2.11 we observe that

$$(4.4) \quad 2^m/r^n \leq 2^m/(m+1) \leq |X| \leq |\text{P}\Gamma\text{L}(n, r)| \leq r^{n^2},$$

and so

$$(4.5) \quad g_n(r) := \frac{\frac{r^n-1}{r-1}}{\log(r)} \leq \frac{n^2+n}{\log(2)}.$$

By first considering the case $r = 2$, we deduce from (4.5) that $n \leq 4$. Now, for a fixed n , the function $g_n(r)$ is increasing for $r \geq 3$. Thus $g_n(3) \leq (n^2+n)/\log(2)$, from which we deduce that $n \leq 3$. By letting $n = 2$ or 3 in (4.4) and using $|\text{P}\Gamma\text{L}(n, r)|$ as an upper bound, we find that $r \leq 16$ or 4 respectively. Recalling that $m \geq 10$, it follows that r, n are as in one of the columns of Table 3.

Consider column 1, so $X \cong \text{PSL}(4, 2)$. In this case $X_1 \cong \text{AGL}(3, 2)$ is perfect contradicting Lemma 4.2. Now consider column 2, so $n = 3$ and $r \in \{3, 4\}$. Consequently, $m = 13$ or 21 . As $\text{P}\Gamma\text{L}(3, r)$ is not 3-transitive, it follows that X is not 3-transitive, and therefore, by [18], is not 3-homogeneous. Thus Proposition 2.7 implies that $\delta \leq 6$. By Lemma 2.18, binary completely regular codes with these parameters for $m = 13$ do not exist. Therefore $(r, m) = (4, 21)$. Since 21 is not a prime power it follows that X_α is a 2-transitive almost simple subgroup of X and therefore X_α contains $\text{PSL}(3, 4)$. Hence $|C| = |X : X_\alpha| \leq 6$. However, Lemma 2.17 implies that $|C| \geq m + 1 = 22$. Thus column 2 does not hold. In column 3, $n = 2$ with $r \leq 16$ and $m = r + 1$, and because $m \geq 10$, it follows that $r = 9, 11, 13$ or 16 . Since X_α is 2-homogeneous, we deduce in each case that X_α is 2-transitive of degree $r + 1$ [18]. For these values of r , every 2-transitive subgroup of degree $r + 1$ of $\text{P}\Gamma\text{L}(2, r)$ contains $\text{PSL}(2, r)$, and so $\text{PSL}(2, r) \lesssim X_\alpha \leq X \lesssim \text{P}\Gamma\text{L}(2, r)$. Hence $|C| = |X : X_\alpha|$ divides $|X : \text{PSL}(2, r)|$ which divides $4, 2, 2, 4$

for $r = 9, 11, 13, 16$ respectively. However, Lemma 2.17 implies that $|C| \geq m + 1 = r + 2$, which is a contradiction in each case. \square

5. PROOF OF THEOREM 1.1

Let C be an X -completely transitive code in $H(m, q)$ with $\delta \geq 5$ and $X \cap \mathfrak{B} = 1$. Firstly suppose that X does not contain A_m . Furthermore, suppose that $|C| > 2$, so C satisfies Hypothesis 4.1. By Corollary 2.10, X is 2-transitive, so X is either of affine or almost simple type. However, it follows from Propositions 4.3 and 4.4 that no such code exists. Thus $|C| = 2$, which by Lemma 2.15 holds if and only if $\delta = m$. Therefore, by Proposition 3.3, $X \cong S_m$ which is a contradiction. Therefore $A_m \lesssim X$. Consequently Proposition 3.2 implies that C is equivalent to the binary repetition code $\text{Rep}(m, 2)$, and that $X \cong S_m$ and $X_\alpha \cong A_m$.

Conversely suppose C is equivalent to $\text{Rep}(m, 2)$ with $m \geq 5$. We saw in Example 2.6 that $\text{Rep}(m, 2)$ is X -completely transitive with $X \cap \mathfrak{B} = 1$, $X \cong S_m$ and $X_\alpha \cong A_m$. As C is equivalent to $\text{Rep}(m, 2)$ there exists $y \in \text{Aut}(\Gamma)$ such that $\text{Rep}(m, 2)^y = C$, and therefore C has minimum distance $\delta = m$. Moreover, by Remark 2.4, C is X^y -completely transitive. Since $X^y \cap \mathfrak{B} = 1$ if and only if $X \cap \mathfrak{B} = 1$, we have that $|C| \geq 2$, $X^y \cap \mathfrak{B} = 1$ and $m = \delta \geq 5$ satisfying the required conditions of Theorem 1.1.

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